calculated from the corresponding values of $\bar{\gamma}_e$, $\bar{\nu}$, and n as deduced from the two latest experiments.[1]

X. CONCLUSIONS

The results of the experiment have been analyzed by expressing the probability of seeing the observed data as a function of the assumed $\mu \rightarrow e+\gamma$ branching ratio B. Limits on the electron range and electron-gamma angle have been used in the selection of events to exclude regions strongly favoring $\mu \rightarrow e+\gamma+\nu+\bar{\nu}$ over $\mu \rightarrow e+\gamma$. Once events with the shortest range electrons have been excluded, the results are substantially independent of the exact choice of limits. The probability is largest for $B=0$, drops to less than 50% of this value for $B=0.8\times10^{-8}$, and to less than 10% of this value for $B=2.2\times10^{-8}$.

The results of this and the two preceding experiments could be combined by multiplication of the three probability distributions. However, this joint probability is not significantly lower for $B\leq 2\times10^{-8}$ and thus does not change the above results appreciably.

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Uncrossed Ladder Graphs in the Feinberg-Pais Theory of Leptonic Weak Interactions

YIH PWU†
University of Pennsylvania, Philadelphia, Pennsylvania

AND

TAI TSUN WU*
Institute for Advanced Study, Princeton, New Jersey

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The Feinberg-Pais theory of summing the uncrossed ladder graphs is re-examined. A single regularization of the W propagator is used throughout. Instead of the iterative procedure used previously, the leading terms of the individual graphs are summed exactly. The result of Feinberg and Pais for energies much below 300 BeV is obtained for all energies in the limit of infinite regulator mass.

1. INTRODUCTION

RECENTLY, Feinberg and Pais[1] studied the higher order effects in weak interactions. In their work, the weak interactions are assumed to be mediated by massive bosons W of spin 1, and neutral lepton currents are assumed to be absent in the Lagrangian. In studying two-body scattering processes involving only leptons, Feinberg and Pais restricted their consideration of higher order weak interactions to the uncrossed ladder graphs only. The rungs of these graphs consist alternatively of W^+ and W^-, a fact of great importance in their work. So far as leptonic weak interactions are concerned, one of the important conclusions of Feinberg and Pais is that, for the so-called allowed processes, the factor

$$-i(\delta_{\mu\nu}+m^{-2}q_\mu q_\nu)/(q^2+m^2) \tag{1.1}$$

in the expression for the matrix element should be replaced, when higher order effects are taken into account, by

$$-i\tfrac{3}{4}[\delta_{\mu\nu}(1-\tfrac{1}{3}m^{-2}q^2)+\tfrac{4}{3}m^{-2}q_\mu q_\nu]/(q^2+m^2) \tag{1.2}$$

provided that q satisfies

$$|q^2g^2/m^2|\ll 1. \tag{1.3}$$

Here m denotes the mass of W, and g is the W-lepton coupling constant.

It is the purpose of this paper to study in more detail the properties of these uncrossed ladder graphs. We do not inquire into the effects of more complicated graphs; instead, given these uncrossed ladder graphs and some rules of computation to be outlined in Sec. 2, we ask what mathematical deductions are possible. Since it

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* Alfred P. Sloan Foundation Fellow. Permanent address: Harvard University, Cambridge, Massachusetts.
[1] G. Feinberg and A. Pais, Phys. Rev. 131, 2724 (1963).

seems necessary to use at least one regularized propagator, we choose as our starting point the ξ-limiting process of Lee and Yang.[2] In other words, we shall regularize the W propagator once while leaving the lepton propagators unregularized.

The Bethe-Salpeter integral equation is simplified in Sec. 3, and the approximate integral equation used by Feinberg and Pais is studied in Sec. 4. In Sec. 5 we sum exactly the leading terms of the uncrossed ladder graphs by first reducing the problem to a pair of differential equations. Without using the iterative procedure of Feinberg and Pais, it is then found that from the differential equations the result (1.2) follows. Moreover, when the present procedure is followed, (1.2) is found to hold even without the restriction (1.3). Therefore, within the approximation of taking into account only the uncrossed ladder graphs, the precise result to the order g^2 is now known. There are, of course, higher order terms in g^2 due to uncrossed ladder graphs,[3] but these are not discussed here. The notation of Feinberg and Pais is followed in this paper.

2. PROCEDURE OF COMPUTATION

We are interested in the dominating part as $\xi \to 0$ of each ladder graph. When there is only one rung, this dominating part is simply

$$M_1(q) = -ig^2[\gamma_\mu(1+\gamma_5)]^{(1)}[\gamma_\nu(1+\gamma_5)]^{(2)} \times\left(\delta_{\mu\nu}+\frac{q_\mu q_\nu}{m^2}\right)\frac{1}{q^2+m^2-i\epsilon}. \quad (2.1)$$

When there is more than one rung, this dominating part can be written down explicitly by using the following

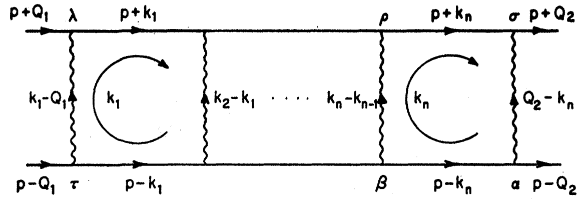

FIG. 1. A scattering graph of order $2(n+1)$.

simplified Feynman rules:

(a) at each W-lepton vertex, insert a factor
$$g\gamma_\mu(1+\gamma_5);$$

(b) for each fermion propagator, insert a factor
$$-i\frac{\not{p}}{p^2-i\epsilon},$$
where $\not{p} = -i\gamma_\mu p_\mu$;

(c) for each W propagator, insert a factor
$$-i\frac{p_\mu p_\nu}{m^2}\left(\frac{1}{p^2-i\epsilon}-\frac{1}{p^2+\Lambda^2-i\epsilon}\right),$$
where $\Lambda^2 \equiv \xi^{-1}m^2$; and

(d) the momenta of external leptons are taken to be zero.

In rule (b) we have neglected the lepton masses, in rule (c) we have neglected the W mass in the denominator. We shall justify these approximations as well as rule (d) by the following considerations.

Take a ladder graph of order $2(n+1)$ as shown in Fig. 1. The contribution from such a graph, before any approximation has been made, is given by (the spinor factors for the external lines are omitted)

$$\begin{aligned}
M_{n+1}(p,Q_1,Q_2) = {} & g^{2(n+1)}\frac{i^{n-1}}{(2\pi)^{4n}}\int d^4k_1\cdots d^4kn \\
& \times\left[\gamma_\sigma(1+\gamma_5)\frac{\not{p}+\not{k}_n+m_l}{(p+k_n)^2+m_l^2-i\epsilon}\gamma_\rho(1+\gamma_5)\frac{\not{p}+\not{k}_{n-1}+m_l}{(p+k_{n-1})^2+m_l^2-i\epsilon}\cdots\frac{\not{p}+\not{k}_1+m_l}{(p+k_1)^2+m_l^2-i\epsilon}\gamma_\lambda(1+\gamma_5)\right]^{(1)} \\
& \times\left[\gamma_\alpha(1+\gamma_5)\frac{\not{p}-\not{k}_n+m_l}{(p-k_n)^2+m_l^2-i\epsilon}\gamma_\beta(1+\gamma_5)\frac{\not{p}-\not{k}_{n-1}+m_l}{(p-k_{n-1})^2+m_l^2-i\epsilon}\cdots\frac{\not{p}-\not{k}_1+m_l}{(p-k_1)^2+m_l^2-i\epsilon}\gamma_\tau(1+\gamma_5)\right]^{(2)} \\
& \times\left[\delta_{\sigma\alpha}+\frac{(Q_2-k_n)_\sigma(Q_2-k_n)_\alpha}{m^2}\right]\left[\frac{1}{(Q_2-k_n)^2+m^2-i\epsilon}-\frac{1}{(Q_2-k_n)^2+\Lambda^2-i\epsilon}\right] \\
& \times\left[\delta_{\rho\beta}+\frac{(k_n-k_{n-1})_\rho(k_n-k_{n-1})_\beta}{m^2}\right]\left[\frac{1}{(k_n-k_{n-1})^2+m^2-i\epsilon}-\frac{1}{(k_n-k_{n-1})^2+\Lambda^2-i\epsilon}\right] \\
& \times\left[\delta_{\lambda\tau}+\frac{(k_1-Q_1)_\lambda(k_1-Q_1)_\tau}{m^2}\right]\left[\frac{1}{(k_1-Q_1)^2+m^2-i\epsilon}-\frac{1}{(k_1-Q_1)^2+\Lambda^2-i\epsilon}\right],
\end{aligned} \quad (2.2)$$

[2] T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962); T. D. Lee, Phys. Rev. **128**, 898 (1962).
[3] See, for example, G. Feinberg and A. Pais, Phys. Rev. **133**, B477 (1964).

where m_l is the lepton mass. To single out the leading term in the variable ξ^{-1}, we make a scale transformation

$$k_i'=k_i/\Lambda\,,\quad i=1\cdots n\,.$$

Since p, Q_1, Q_2, m, and m_l are fixed quantities, p/Λ, m_l/Λ are all small in the limit $\xi\rightarrow 0$ and can be neglected in calculating the leading term, which we shall denote by N_{n+1}:

$$N_{n+1}=\frac{g^2}{m^2}\left(\frac{g\Lambda}{m}\right)^{2n}\frac{i^{n-1}}{(2\pi)^{4n}}\int d^4k_1'\cdots d^4k_n'\left[\gamma_\sigma(1+\gamma_5)\frac{k_n'}{k_n'^2-i\epsilon}\gamma_\rho(1+\gamma_5)\frac{k_{n-1}'}{k_{n-1}'^2-i\epsilon}\cdots\frac{k_1'}{k_1'^2-i\epsilon}\gamma_\lambda(1+\gamma_5)\right]^{(1)}$$
$$\times\left[\gamma_\alpha(1+\gamma_5)\frac{-k_n'}{k_n'^2-i\epsilon}\gamma_\beta(1+\gamma_5)\frac{-k_{n-1}'}{k_{n-1}'^2-i\epsilon}\cdots\frac{-k_1'}{k_1'^2-i\epsilon}\gamma_\tau(1+\gamma_5)\right]^{(2)}k_{n\sigma}'k_{n\alpha}'\left[\frac{1}{k_n'^2-i\epsilon}-\frac{1}{k_n'^2+1-i\epsilon}\right]$$
$$\times(k_n'-k_{n-1}')_\rho(k_n'-k_{n-1}')_\beta\left[\frac{1}{(k_n'-k_{n-1}')-i\epsilon}-\frac{1}{(k_n'-k_{n-1}')^2+1-i\epsilon}\right]\cdots k_{1\lambda}'k_{1\tau}'\left[\frac{1}{k_1'^2-i\epsilon}-\frac{1}{k_1'^2+1-i\epsilon}\right].\quad(2.3)$$

This leading term is thus independent of the external momenta and the lepton masses. And the rules of computation outlined above follow.

As discussed by Feinberg and Pais,[1] the allowed and forbidden processes are, respectively, described by

$$M_{\text{odd}}=M_1(q)+\sum_{n=1}^{\infty}N_{2n+1}\quad(2.4)$$

and

$$M_{\text{even}}=\sum_{n=1}^{\infty}N_{2n}.\quad(2.5)$$

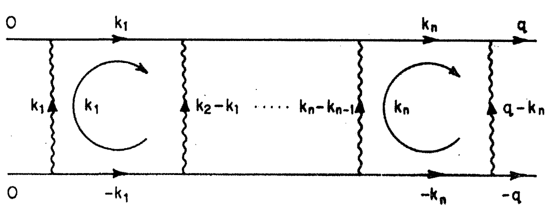

FIG. 2. The graph for $N_{n+1}(q)$.

As $\xi\rightarrow 0$, each N_n is infinite. Thus the limit $\xi\rightarrow 0$ is to be taken after the sums are found. Moreover, for small but finite ξ, the sums are divergent. It is therefore necessary to give a meaning to the divergent series. This point is to be discussed later.

Contrary to the procedure of Feinberg and Pais, in this paper we observe strictly the rule of summing only the leading terms, in the limit $\xi\rightarrow 0$ while all physical quantities are fixed. Since these leading terms $N_n(n>1)$ are independent of momenta and lepton masses, the matrix elements for the allowed and forbidden processes are respectively determined by one real number each, namely the values of the infinite sums in (2.4–5) in the limit $\xi\rightarrow 0$, which may be interpreted as direct four-fermion interactions. In other words, from the present point of view, the discussion near the end of Sec. 7a in the paper of Feinberg and Pais[1] applies for all q, not necessarily small.

3. INTEGRAL EQUATIONS

Let us denote by $N_{n+1}(q)$ the amplitude of a graph with external momenta $(0,0;q,-q)$ as shown in Fig. 2, computed on the basis of the simplified Feynman rules (a)–(c). Thus, for example,

$$N_1(q)=-ig^2[\gamma_\mu(1+\gamma_5)]^{(1)}[\gamma_\nu(1+\gamma_5)]^{(2)}\times\frac{q_\mu q_\nu}{m^2}\left(\frac{1}{q^2-i\epsilon}-\frac{1}{q^2+\Lambda^2-i\epsilon}\right).$$

Define

$$N^{(+)}(q)=\sum_{n=0}^{\infty}N_{n+1}(q)\quad(3.1)$$

and

$$N^{(-)}(q)=\sum_{n=0}^{\infty}(-1)^nN_{n+1}(q).\quad(3.2)$$

The following relations are obtained:

$$N_n=\lim_{q\rightarrow 0}N_n(q)\,,\quad n>1\,,\quad(3.3)$$

$$M_{\text{odd}}=M_1(q)+\tfrac{1}{2}\lim_{q\rightarrow 0}[N^{(+)}(q)+N^{(-)}(q)-2N_1(q)]\quad(3.4)$$

and

$$M_{\text{even}}=\tfrac{1}{2}\lim_{q\rightarrow 0}[N^{(+)}(q)+N^{(-)}(q)].\quad(3.5)$$

Since $N_n(q)$ satisfies the recurrence relation

$$N_{n+1}(q)=ig^2\int d^4k\left[\gamma_\sigma(1+\gamma_5)\frac{k}{k^2-i\epsilon}\right]^{(1)}\left[\gamma_\alpha(1+\gamma_5)\frac{-k}{k^2-i\epsilon}\right]^{(2)}$$
$$\times\frac{(q-k)_\sigma(q-k)_\alpha}{m^2}\left[\frac{1}{(q-k)^2-i\epsilon}-\frac{1}{(q-k)^2+\Lambda^2-i\epsilon}\right]N_n(k)\,,\quad(3.6)$$

it follows that $N^{(\pm)}(q)$ satisfy the following integral equations:

$$N^{(\pm)}(q)=-ig^2\frac{q_\mu q_\nu}{m^2}\left(\frac{1}{q^2-i\epsilon}-\frac{1}{q^2+\Lambda^2-i\epsilon}\right)[\gamma_\mu(1+\gamma_5)]^{(1)}[\gamma_\nu(1+\gamma_5)]^{(2)}$$
$$\pm g^2\frac{i}{(2\pi)^4}\int d^4k\left[\gamma_\sigma(1+\gamma_5)\frac{k}{k^2-i\epsilon}\right]^{(1)}\left[\gamma_\alpha(1+\gamma_5)\frac{-k}{k^2-i\epsilon}\right]^{(2)}\frac{Q_\sigma Q_\alpha}{m^2}\left[\frac{1}{(q-k)^2-i\epsilon}-\frac{1}{(q-k)^2+\Lambda^2-i\epsilon}\right]N^{(\pm)}(k),\quad (3.7)$$

where $Q=q-k$.

The integral equation (3.7) can be simplified in two steps. We shall first eliminate the γ matrices by substituting into (3.7) the explicit dependence of $N^{(\pm)}(q)$ on the γ matrices

$$N^{(\pm)}(q)=[\gamma_\mu(1+\gamma_5)]^{(1)}[\gamma_\nu(1+\gamma_5)]^{(2)}N_{\mu\nu}^{(\pm)}(q)\quad (3.8)$$

and by introducing the notation $\xi_{\lambda\rho\sigma\tau}$ defined by[1]

$$(1-\gamma_5)\gamma_\lambda\gamma_\rho\gamma_\sigma=\xi_{\lambda\rho\sigma\tau}(1-\gamma_5)\gamma_\tau.\quad (3.9)$$

Substituting (3.8) and (3.9) into (3.7) and canceling out the γ matrices, we obtain

$$N_{\mu\nu}^{(\pm)}(q)=-ig^2\frac{q_\mu q_\nu}{m^2}\left(\frac{1}{q^2-i\epsilon}-\frac{1}{q^2+\Lambda^2-i\epsilon}\right)$$
$$\pm\frac{4ig^2}{(2\pi)^4}\int d^4k\frac{k_\rho k_\sigma}{(k^2-i\epsilon)^2}\frac{Q_\lambda Q_\tau}{m^2}\left[\frac{1}{(q-k)^2-i\epsilon}-\frac{1}{(q-k)^2+\Lambda^2-i\epsilon}\right]N_{\alpha\beta}^{(\pm)}(k)\xi_{\lambda\rho\alpha\mu}\xi_{\tau\sigma\beta\nu}.\quad (3.10)$$

Since $N_{\alpha\rho}^{(\pm)}(k)$ is of the form

$$N_{\alpha\beta}^{(\pm)}(k)=A^{(\pm)}(k^2)\delta_{\mu\nu}+B^{(\pm)}(k^2)k_\mu k_\nu\quad (3.11)$$

and $\xi_{\lambda\rho\alpha\mu}$ satisfies

$$k_\rho k_\alpha\xi_{\lambda\rho\alpha\mu}=k^2\delta_{\lambda\mu}\quad (3.12)$$

and

$$\delta_{\alpha\beta}k_\rho k_\sigma Q_\lambda Q_\tau\xi_{\lambda\rho\alpha\mu}\xi_{\tau\sigma\beta\nu}=k^2Q^2\delta_{\mu\nu},\quad (3.13)$$

(3.10) can be further simplified to

$$A^{(\pm)}(q^2)\delta_{\mu\nu}+B^{(\pm)}(q^2)q_\mu q_\nu=-ig^2\frac{q_\mu q_\nu}{m^2}\left(\frac{1}{q^2-i\epsilon}-\frac{1}{q^2+\Lambda^2-i\epsilon}\right)$$
$$\pm\frac{4ig^2}{(2\pi)^4m^2}\int d^4k\left[A^{(\pm)}(k^2)\frac{Q^2}{k^2-i\epsilon}\delta_{\mu\nu}+B^{(\pm)}(k^2)Q_\mu Q_\nu\right]\left(\frac{1}{(q-k)^2-i\epsilon}-\frac{1}{(q-k)^2+\Lambda^2-i\epsilon}\right).\quad (3.14)$$

These integral equations are to be studied further in Sec. 5.

4. FEINBERG-PAIS APPROXIMATION PROCEDURE

The integral equation (3.14) corresponds to the "approximate integral equation" (4.27) of Feinberg and Pais,[1] if we neglect q in Q and thus put $Q=-k$. In this case, the equation is soluble since it takes the form of an algebraic equation in coordinate space. Because of our rule of summing only the leading terms, even with this approximation our calculation differs somewhat from that of Feinberg and Pais. In this section, we study this approximate integral equation in some detail. This seems worthwhile because it makes contact both with the work of Feinberg and Pais and with the consideration of the next section, where explicit solution is not possible. In particular, we shall see the consequence of the singularity at $q^2=0$ of the inhomogeneous term, and also treat a contour integration in a way to bring out the important role played by analytical continuation.

The solution of the approximate integral equation is

$$A^{(\pm)}(q^2)\delta_{\mu\nu}+B^{(\pm)}(q^2)q_\mu q_\nu$$
$$=\frac{ig^2}{m^2}\int d^4x e^{iqx}\partial_\mu\partial_\nu\bar{\Delta}_F(x^2)\left[1\mp\frac{4ig^2}{m^2}\bar{\Delta}_F(x^2)\right]^{-1},\quad (4.1)$$

where

$$\bar{\Delta}_F(x^2)=\frac{1}{(2\pi)^4}\int d^4p e^{-ipx}\left(\frac{1}{p^2-i\epsilon}-\frac{1}{p^2+\Lambda^2-i\epsilon}\right).\quad (4.2)$$

We shall study these solutions in some detail in this section.

We compute the trace of (4.1):

$$4A^{(\pm)}(q^2)+B^{(\pm)}(q^2)q^2 =\frac{ig^2}{m^2}\int d^4x e^{iqx}\Box\bar{\Delta}_F(x^2)\left[1\mp\frac{4ig^2}{m^2}\bar{\Delta}_F(x^2)\right]^{-1}. \quad (4.3)$$

Explicitly,

$$\bar{\Delta}_F(x^2)=\frac{i}{4\pi^2x^2}-\frac{i\Lambda}{4\pi^2\sqrt{x^2}}K_1(\Lambda\sqrt{x^2})\theta(x^2) +\frac{\Lambda}{8\pi\sqrt{-x^2}}H_1^{(2)}(\Lambda\sqrt{-x^2})\theta(-x^2). \quad (4.4)$$

Moreover,

$$\Box\bar{\Delta}_F(x^2)=-\Lambda^2\Delta_F(x^2), \quad (4.5)$$

where $\Delta_F(x^2)$ is defined as

$$\Delta_F(x^2)=\frac{1}{(2\pi)^4}\int d^4p e^{-ipx}\frac{1}{p^2+\Lambda^2-i\epsilon}. \quad (4.6)$$

We have, as shown in Appendix A,

$$\Delta_F(x^2)=\frac{i}{4\pi^2}\frac{1}{x^2+i\epsilon} -\left[\frac{i}{4\pi^2x^2}-\frac{i\Lambda}{4\pi^2\sqrt{x^2}}K_1(\Lambda\sqrt{x^2})\right]\theta(x^2) -\left[\frac{i}{4\pi^2x^2}+\frac{\Lambda}{8\pi\sqrt{-x^2}}H_1^{(2)}(\Lambda\sqrt{-x^2})\right]\theta(-x^2). \quad (4.7)$$

Using the reduction formula of Feinberg and Pais,[1] we can write the trace (4.3) as the sum of a Bessel integral and a contour integral over the complex variable η defined by $\eta^2=x^2$. That is

$$4A^{(\pm)}(q^2)+B^{(\pm)}(q^2)q^2=J^{(\pm)}(q^2)+C^{(\pm)}(q^2), \quad (4.8)$$

where, for $q^2>0$,

$$J^{(\pm)}(q^2)=\frac{ig^2}{m^2}\frac{4i\pi^2}{q}\int_0^\infty J_1(q\eta)\Lambda^2\Delta_F(\eta) \times\left[1\mp\frac{4ig^2}{m^2}\bar{\Delta}_F(\eta)\right]^{-1}\eta^2d\eta, \quad (4.9)$$

and

$$C^{(\pm)}(q^2)=-\frac{ig^2}{m^2}\frac{4i\pi^2}{q}\int_C H_1^{(1)}(q\eta)\Lambda^2\Delta_F(\eta) \times\left[1\mp\frac{4ig^2}{m^2}\bar{\Delta}_F(\eta)\right]^{-1}\eta^2d\eta. \quad (4.10)$$

Here we use

$$\Delta_F(\eta)=\frac{i}{4\pi^2}\left\{\frac{1}{\eta^2+i\epsilon}-\left[\frac{1}{\eta^2}-\frac{\Lambda}{\eta}K_1(\Lambda\eta)\right]\right\}, \quad (4.11)$$

$$\bar{\Delta}_F(\eta)=\frac{i}{4\pi^2}\left[\frac{1}{\eta^2}-\frac{\Lambda}{\eta}K_1(\Lambda\eta)\right] \quad (4.12)$$

and the contour is along the first quadrant counterclockwise, at least for sufficiently small Λ^2.

Let $\zeta=\Lambda\eta$. The Bessel integrals (4.9) become, in the limit $q\to 0$,

$$J^{(\pm)}(0)=-\frac{ig^2}{2m^2}\int_0^\infty K_1(\zeta) \times\left\{1\pm\frac{g^2\Lambda^2}{\pi^2m^2}\left[\frac{1}{\zeta^2}-\frac{1}{\zeta}K_1(\zeta)\right]\right\}^{-1}\zeta^2d\zeta, \quad (4.13)$$

which approach the limit zero as $\Lambda^2\to\infty$. On the other hand, in terms of the complex variable ζ, the contour integrals (4.10) can be rewritten as

$$C^{(\pm)}(q^2)=\frac{ig^2}{m^2q}\int_C H_1^{(1)}(q\zeta\Lambda^{-1}) \times\left\{\frac{1}{\zeta^2+i\epsilon}-\left[\frac{1}{\zeta^2}-\frac{1}{\zeta}K_1(\zeta)\right]\right\} \times\left\{1\pm\frac{g^2\Lambda^2}{\pi^2m^2}\left[\frac{1}{\zeta^2}-\frac{1}{\zeta}K_1(\zeta)\right]\right\}^{-1}\zeta^2d\zeta, \quad (4.14)$$

which are also equal to zero for any space-like q in the limit $\Lambda^2\to\infty$. To see this, we first notice that the common numerator of the integrals is analytic in the first quadrant of the complex ζ plane for any space-like q which can be chosen as small as we please. In Appendix B, we study the locations of the zeros of the denominators.

$$D^{(\pm)}(\zeta,Z)=1\pm Z\left[\frac{1}{\zeta^2}-\frac{1}{\zeta}K_1(\zeta)\right], \quad (4.15)$$

where

$$Z=\frac{g^2\Lambda^2}{\pi^2m^2} \quad (4.16)$$

and obtain the following conclusions: There are no zeros of $D^{(+)}(\zeta,Z)$ in the first quadrant if $Z\ll 1$. When Z increases continuously, the zeros of $D^{(+)}(\zeta,Z)$ move into the first quadrant from the left across the positive imaginary axis. The denominator $D^{(-)}(\zeta,Z)$ has, in addition to the zeros which moves across the positive imaginary axis, a zero which moves from left to right on the positive real axis and which goes as $Z^{1/2}$ when $Z\to\infty$. Therefore we shall take different contours $C_{(\pm)}$ for the integrals $C^{(\pm)}(q^2)$. For a large value of Z, $C_{(\pm)}$ are

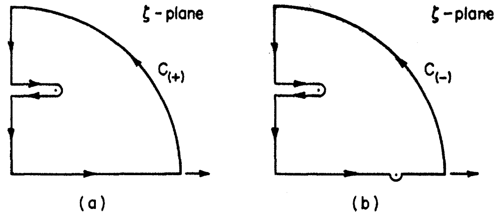

FIG. 3. Contours of integration for $C^{(\pm)}(q)$.

shown in Figs. 3(a) and 3(b), respectively. This choice of $C_{(-)}$ is that of Feinberg and Pais.[1] Then we have, for space-like q, $C^{(+)}(q^2)=0$ for all Z, and $C^{(-)}(q^2)$, being equal to the residue at pole on the real axis, tends to zero as $Z\to\infty$. As a consequence, we have

$$\lim_{\Lambda\to\infty}\lim_{q\to 0}[4A^{(\pm)}(q^2)+B^{(\pm)}(q^2)q^2]=0. \qquad (4.17)$$

Note that the order of taking limiting values is the one prescribed in Sec. 2.

We now turn to calculating the function $B^{(\pm)}(q^2)$ as $q\to 0$. Let us define $N_{n\mu\nu}$, $N_{n\mu\nu}(q)$ by

$$N_n=[\gamma_\mu(1+\gamma_5)]^{(1)}[\gamma_\nu(1+\gamma_5)]^{(2)}N_{n\mu\nu}, \qquad (4.18)$$

and

$$N_n(q)=[\gamma_\mu(1+\gamma_5)]^{(1)}[\gamma_\nu(1+\gamma_5)]^{(2)}N_{n\mu\nu}(q), \qquad (4.19)$$

and $A_n(q^2)$, $B_n(q^2)$ by

$$N_{n\mu\nu}(q^2)=A_n(q^2)\delta_{\mu\nu}+B_n(q^2)q_\mu q_\nu. \qquad (4.20)$$

Because of the relations (3.3), the limit

$$\lim_{q\to 0}N_{n\mu\nu}(q^2)$$

exists for each $n>1$ and is equal to $N_{n\mu\nu}$ which has the form $C_n\delta_{\mu\nu}$, where C_n is a constant. In other words

$$C_n\delta_{\mu\nu}=\lim_{q\to 0}[A_n(q^2)\delta_{\mu\nu}+B_n(q^2)q_\mu q_\nu], \quad n>1. \qquad (4.21)$$

Then it follows that, for $n>1$,

$$C_n=A_n(0) \quad\text{and}\quad \lim_{q\to 0}B_n(q^2)q^2\to 0,$$

since the components q approach zero independently. Therefore, if the limit $q\to 0$ is taken termwise

$$\lim_{q\to 0}B^{(\pm)}(q^2)q^2=\lim_{q\to 0}B_1^{(\pm)}(q^2)q^2=-\frac{ig^2}{m^2}. \qquad (4.22)$$

The limit (4.22) is also obtained from a direct calculation of (4.1) in Appendix C.

From (4.17) and (4.22), we can write

$$\lim_{\Lambda\to\infty}\lim_{q\to 0}A^{(\pm)}(q^2)=\frac{ig^2}{4m^2}, \qquad (4.23)$$

and consequently

$$N_{\mu\nu}^{(\pm)}(q)=\frac{ig^2}{m^2}\left[\tfrac{1}{4}\delta_{\mu\nu}-\frac{1}{q^2}q_\mu q_\nu\right]+O(q^2). \qquad (4.24)$$

The correction from higher order ladder graphs is

$$\tfrac{1}{2}\lim_{q\to 0}[N^{(+)}(q)+N^{(-)}(q)-2N_1(q)]_{\mu\nu}=\frac{ig^2}{4m^2}\delta_{\mu\nu}, \qquad (4.25)$$

and the amplitude of the allowed process becomes

$$M_{\text{odd}}=-ig^2[\gamma_\mu(1+\gamma_5)]^{(1)}[\gamma_\nu(1+\gamma_5)]^{(2)} \times[\delta_{\mu\nu}(\tfrac{3}{4}-\tfrac{1}{4}m^{-2}q^2)+m^{-2}q_\mu q_\nu]/(q^2+m^2-i\epsilon). \qquad (4.26)$$

We have thus reproduced the result of Feinberg and Pais[1] without trying to justify the approximation.

5. SUMMING THE LADDER GRAPHS

In this section we study the problem of summing the uncrossed ladder graphs without using the Feinberg-Pais approximation. The starting point is the integral equation (3.14). Let

$$\alpha^{(\pm)}(x^2)=(2\pi)^{-4}\int d^4q e^{-iqx}(q^2-i\epsilon)^{-1}A^{(\pm)}(q^2), \qquad (5.1)$$

and

$$B^{(\pm)}(x^2)=(2\pi)^{-4}\int d^4q e^{-iqx}B^{(\pm)}(q^2), \qquad (5.2)$$

then in coordinate space (3.14) takes the form

$$\Box\alpha^{(\pm)}(x^2)\delta_{\mu\nu}+\partial_\mu\partial_\nu B^{(\pm)}(x^2) =-ig^2m^{-2}\partial_\mu\partial_\nu\bar{\Delta}_F(x^2)\pm 4ig^2m^{-2}[\alpha^{(\pm)}(x^2)\Box\bar{\Delta}_F(x^2)\delta_{\mu\nu} +B^{(\pm)}(x^2)\partial_\mu\partial_\nu\bar{\Delta}_F(x^2)], \qquad (5.3)$$

which is a differential equation. Among the multitude of solutions of this differential equation, we choose the one such that (1) $A^{(\pm)}(0)$ exists, (2) $B^{(\pm)}(q^2)=O(q^{-2})$ for small q^2 and (3) the integral in (3.14) exists.

We again use the complex variable η. In view of (4.9) and (4.10), it is sufficient to consider η to be in the first quadrant. It follows from (5.3) that $B^{(\pm)}(x^2)$ satisfies

$$\left(\frac{d^2}{d\eta^2}-\frac{1}{\eta}\frac{d}{d\eta}\right)B^{(\pm)} =-i\frac{g^2}{m^2}\left[\left(\frac{d^2}{d\eta^2}-\frac{1}{\eta}\frac{d}{d\eta}\right)\bar{\Delta}_F\right](1\mp 4B^{(\pm)}). \qquad (5.4)$$

It is convenient to take the trace of (5.3). If we define

$$T^{(\pm)}(x^2)=4\alpha^{(\pm)}(x^2)+B^{(\pm)}(x^2), \qquad (5.5)$$

then $T^{(\pm)}(x^2)$ satisfies

$$\left(\frac{d^2}{d\eta^2}+\frac{3}{\eta}\frac{d}{d\eta}\right)T^{(\pm)} =-i\frac{g^2}{m^2}\left[\left(\frac{d^2}{d\eta^2}+\frac{3}{\eta}\frac{d}{d\eta}\right)\bar{\Delta}_F\right](1\mp 4T^{(\pm)}). \qquad (5.6)$$

Equation (5.6) can be simplified by using (4.5)

$$\left(\frac{d^2}{d\eta^2}+\frac{3}{\eta}\frac{d}{d\eta}\right)T^{(\pm)}=i\frac{g^2\Lambda^2}{m^2}\Delta_F(1\mp 4T^{(\pm)}). \quad (5.7)$$

The problem is to determine $A(0)$ from (5.4) and (5.7) together with appropriate boundary conditions. By (4.7), (5.4) and (5.7) have no singularity in the first quadrant of the complex η plane, and thus both $B^{(\pm)}$ and $T^{(\pm)}$ are analytic there. Hence, the contour integration part of the reduction formula of Feinberg and Pais[1] is zero in each case. Therefore, it is sufficient to restrict η to be real and positive. It is interesting to note that this part of the argument is much simpler here than in the last section.

We first consider (5.7) in some detail. Since $\eta>0$, we can set $\epsilon\to 0$ in (4.7), and (5.7) reduces simply to

$$\left(\frac{d^2}{d\eta^2}+\frac{3}{\eta}\frac{d}{d\eta}\right)T^{(\pm)}=-Z\frac{\Lambda}{\eta}K_1(\Lambda\eta)(\tfrac{1}{4}\mp T^{(\pm)}), \quad (5.8)$$

where Z is defined by (4.16). Consider $T^{(\pm)}$ to be a function of η and Z. In Appendix D it is shown that the iterative solution of (5.8) with the boundary conditions (D3) and (D4) converges for each $\eta>0$ provided that

$$|Z|<1. \quad (5.9)$$

This convergent solution defines $T^{(\pm)}$ when (5.9) is satisfied, and we define $T^{(\pm)}$ in general by analytical continuation from this solution. In this process of analytical continuation, Λ is taken to have a small negative imaginary part. This gives a definitive meaning to the divergent series in the present case.

If a solution of the corresponding homogeneous equation is assumed to be of the form η^λ, then the indicial equations give the following values for λ:

as $\eta\to\infty$

$$\lambda=0,\ -2; \quad (5.10)$$

as $\eta\to 0$, then for $T^{(+)}$

$$\lambda=-1\pm(1+Z)^{1/2}, \quad (5.11)$$

and for $T^{(-)}$

$$\lambda=-1\pm(1-Z)^{1/2}. \quad (5.12)$$

When $g\to 0$, $g^{-2}T^{(\pm)}$ may be found from the lowest order graph:

$$\lim_{g\to 0}g^{-2}T^{(\pm)}=-im^{-2}\bar{\Delta}_F. \quad (5.13)$$

Since $T^{(\pm)}$ is analytic in Z within the unit circle, (5.13) rules out one of the solutions in each of the cases (5.10–12). Accordingly, we get the following behavior for $T^{(\pm)}$ when $Z<1$:

as $\eta\to\infty$,

$$T^{(\pm)}\sim\text{const}\,\eta^{-2} \quad (5.14)$$

and as $\eta\to 0$,

$$T^{(\pm)}\mp\tfrac{1}{4}\sim\text{const}\,\eta^{-1+\sqrt{(1\pm Z)}}. \quad (5.15)$$

Equations (5.8), (5.14), and (5.15) determines $T^{(\pm)}$ uniquely when $Z<1$.

Rewrite (5.14) in the form

$$T^{(\pm)}=a_0^{(\pm)}\eta^{-2}+o(\eta^{-2}). \quad (5.16)$$

Then $a_0^{(\pm)}$ is a function of Z. We now note that, since (5.8), (5.14), and (5.15) also determine a unique $T^{(\pm)}$ even outside the unit circle, so long as $Z\neq\mp 1$, respectively, they can be used directly to determine $a_0^{(\pm)}$. Furthermore, this gives the proper analytical continuation.

Since Λ is taken to have a small negative imaginary part, for $Z<1$ (5.15) is replaced by

$$T^{(+)}\sim\eta^{-1+\sqrt{(1+Z)}} \quad (5.17)$$

and

$$T^{(-)}\sim\eta^{-1+i\sqrt{(Z-1)}} \quad (5.18)$$

as $\eta\to 0$. Assume $Z\to\infty$ and we want to determine $a_0^{(\pm)}$ from (5.8) and (5.16–18). This problem can be solved by the WKB procedure.

When $Z\zeta^{-1}K_1(\zeta)\gg 1$, it follows from (5.8) and (5.17–18) that

$$T^{(+)}\sim\tfrac{1}{4}+C_1^{(+)}\zeta^{-1}[\zeta K_1(\zeta)]^{-1/4}\times\exp\left\{Z^{1/2}\int d\zeta[\zeta^{-1}K_1(\zeta)]^{1/2}\right\}, \quad (5.19)$$

and

$$T^{(-)}\sim-\tfrac{1}{4}+C_1^{(-)}\zeta^{-1}[\zeta K_1(\zeta)]^{-1/4}\times\exp\left\{iZ^{1/2}\int d\zeta[\zeta^{-1}K_1(\zeta)]^{1/2}\right\}. \quad (5.20)$$

On the other hand, when $Z\zeta^{-1}K_1(\zeta)\ll 1$, (5.16) holds. It remains to consider the transition region defined by

$$|1-\zeta/\zeta_0|\ll 1, \quad (5.21)$$

where ζ_0 satisfies

$$Z\zeta_0^{-1}K_1(\zeta_0)=1. \quad (5.22)$$

When Z is large, so is ζ_0. Thus, in this region, (5.8) may be approximated by

$$d^2T^{(\pm)}/d\zeta^2=-e^{-(\zeta-\zeta_0)}(\tfrac{1}{4}\mp T^{(\pm)}). \quad (5.23)$$

Equation (5.23) can be solved in terms of Bessel functions. By (5.19–20) the appropriate solutions are

$$T^{(+)}=\tfrac{1}{4}+C_2^{(+)}K_0(2e^{-(\zeta-\zeta_0)/2}), \quad (5.24)$$

and

$$T^{(-)}=-\tfrac{1}{4}-\tfrac{1}{2}\pi iC_2^{(-)}H_0^{(2)}(2e^{-(\zeta-\zeta_0)/2}). \quad (5.25)$$

The relation between $C_1^{(\pm)}$ and $C_2^{(\pm)}$ is of no interest here. Let $\zeta-\zeta_0\gg 1$, then (5.24) and (5.25) reduce to

$$T^{(\pm)}\sim\pm\tfrac{1}{4}+C_2^{(\pm)}\tfrac{1}{2}(\zeta-\zeta_0). \quad (5.26)$$

On the other hand, when (5.21) holds, (5.16) gives

$$T^{(\pm)}\sim a_0^{(\pm)}\Lambda^2\zeta_0^{-2}[1-2\zeta_0^{-1}(\zeta-\zeta_0)]. \quad (5.27)$$

A comparison of (5.27) with (5.26) gives approximately

$$a_0^{(\pm)}=\pm\tfrac{1}{4}\zeta_0^2\Lambda^{-2}. \quad (5.28)$$

Since ζ_0 is defined by (5.22), it follows from (5.28) that

$$a_0^{(\pm)}=0 \quad (5.29)$$

as $\Lambda\rightarrow\infty$. In momentum space, (4.17) follows immediately from (5.29).

By an argument entirely analogous to that given in the last section, we again get (4.22) provided that it is permissible to take limits termwise. From the point of view of the differential equation (5.4), if a solution of the corresponding homogeneous equation is assumed to have the form η^λ as $\eta\rightarrow\infty$, then the indicial equation gives

$$\lambda=0,2. \quad (5.30)$$

But from the second requirement under (5.3), $B^{(\pm)}$ must have the following behavior

$$B^{(\pm)}(x^2)\sim a_0'^{(\pm)}\eta^{-2} \quad (5.31)$$

as $\eta\rightarrow\infty$. To determine $a_0'^{(\pm)}$, it is sufficient to study (5.4) for large positive η:

$$\left(\frac{d^2}{d\eta^2}-\frac{1}{\eta}\frac{d}{d\eta}\right)B^{(\pm)}=\frac{2g^2}{\pi^2m^2}(1\mp4B^{(\pm)})\eta^{-4}. \quad (5.32)$$

Therefore

$$a_0'^{(\pm)}=\frac{g^2}{4\pi^2m^2}, \quad (5.33)$$

at least for sufficient small Z. By analytical continuation, (5.33) holds for all Z and (4.22) follows.

Once (4.17) and (4.22) are obtained, the rest of the computation is identical with that of Sec. 4.

6. DISCUSSION

By summing exactly the leading terms of the uncrossed ladder graphs, we have verified one of the important conclusions of Feinberg and Pais,[1] as expressed by (1.2). Our result is more general than theirs in that we do not need the restriction (1.3). However, this generality is mostly illusory. On the one hand, (1.3) is satisfied for almost all energies available in the laboratory at the present time; and, on the other hand, (1.2) must break down for sufficiently high energies because it violates unitarity.

Since the final answer is identical to that of Feinberg and Pais when (1.3) is satisfied, all their discussions, with the exception of the part pertaining to very high energies only, still hold without alteration. However, the following remark may be appropriate here. Although, in general, higher order weak interactions may lead to a violation of local action of lepton pairs, no such violation occur within the present approximation. This is simply a consequence of the fact that, within the present approximation, the result takes the form of a sum of two terms, one of which is just the usual one through an intermediate boson, while the other one represents a direct four-fermion interaction as given by (4.25).

The procedure followed may be summarized in terms of the following steps: (1) take the uncrossed ladder graphs; (2) compute the term containing the highest power of ξ^{-1} for each of these graphs; (3) sum these leading terms; and (4) take the limit $\xi\rightarrow0$. It may be worth re-emphasizing that in step 2, with the exception of the lowest order graph, there is no momentum dependence. Accordingly, the quantities $N_n(q)$ must be regarded as devoid of physical meaning for $q\neq0$, and they are introduced for mathematical convenience only. Thus, even though the integral equation takes the Bethe-Salpeter form, the usual physical interpretations of Bethe-Salpeter equations can be applied only with extreme caution. In particular, to extract physical information, it is necessary to take the limit $q\rightarrow0$ while still keeping ξ finite. There are two obvious ways of taking this limit; we can either sum over the various graphs first and then take the limit, or take the limit $q\rightarrow0$ for each graph and then sum these limiting values. It is perhaps satisfying to find that at least these two ways of taking the limit give the same answer.

The series of contributions from the ladder graphs under consideration may be divergent. When the approximation procedure of Feinberg and Pais[1] is employed, it may be seen from (4.1) that the series is divergent for all values of ξ. Even though there is still a natural way of giving a meaning to this sum, such a situation does not seem to have been encountered before in physics. When the uncrossed ladder graphs are summed as in Sec. 5, not only do we manage to avoid the approximation procedure of Feinberg and Pais together with its numerous related questions, but we also find that the series is indeed convergent when (5.9) is satisfied. Consequently, it is possible to assign a unique value by analytical continuation to the series, which is divergent when ξ is sufficiently small. The situation is accordingly a rather familiar one in physics.

The most serious problem with the present calculation, as well as that of Feinberg and Pais, is whether any consideration neglecting all graphs but the uncrossed ladder graphs is of any relevence to the physics of weak interactions. In particular, we may ask the much more restrictive question what happens if the single rung of the uncrossed ladder is replaced by a more complicated graph. So far it can only be said that (1) the independence of the leading term on momenta and lepton masses holds in general, provided that the graph is convergent; and (2) the trace T still seems to play an important role, at least in the restrictive case stated above.

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APPENDIX A

The expression (4.7) is obtained by substituting into the standard expression for $\Delta_F(x^2)$

$$\Delta_F(x^2)=\frac{1}{4\pi}\delta(x^2)+\frac{i\Lambda}{4\pi^2\sqrt{x^2}}K_1(\Lambda\sqrt{x^2})\theta(x^2)-\frac{\Lambda}{8\pi\sqrt{-x^2}}H_1^{(2)}(\Lambda\sqrt{-x^2})\theta(-x^2)\,, \quad \text{(A1)}$$

the well-known identity

$$\frac{1}{x^2+i\epsilon}=\frac{P}{x^2}-\pi i\delta(x^2)\,, \quad \text{(A2)}$$

where P denotes the principal value.

We shall show that $\Delta_F(q^2)=(q^2+\Lambda^2-i\epsilon)^{-1}$, if we substitute the expression (4.7) into the reduction formula of Feinberg and Pais. For $q^2>0$

$$\Delta_F(q^2)=\frac{4i\pi^2}{q}\left[\int_C H_1^{(1)}(q\eta)\Delta_F(\eta)\eta^2d\eta-\int_0^\infty J_1(q\eta)\Delta_F(\eta)\eta^2d\eta\right], \quad \text{(A3)}$$

where $\Delta_F(\eta)$ is given in (4.11) and the contour is along the first quadrant counter-clockwise. The contour integral is zero. And the Bessel integral gives the desired result by the Weber-Schafheitlin formula.[4]

APPENDIX B

We study the motion of the zeros of $D^{(\pm)}(\zeta,Z)$ defined by (4.15) as Z goes from 0 to ∞. We concentrate only on the first quadrant.

When $Z=0$, $D^{(\pm)}(\zeta,0)=1$. There are no zeros in the first quadrant of the complex ζ plane.

When $Z\ll1$, it is sufficient to let $|\zeta|\ll1$ in the equation $D^{(\pm)}(\zeta,Z)=0$. In this case,

$$D^{(\pm)}(\zeta,Z)\sim1\pm\frac{Z}{2}\ln\frac{1}{\zeta}\,. \quad \text{(B1)}$$

There is no solution for $D^{(+)}=0$, while $D^{(-)}$ has a zero at $e^{-2/Z}$ on the real axis.

As Z increases, zeros of $D^{(\pm)}(\zeta,Z)$ may enter into the first quadrant through the walls: (a) the positive real axis, (b) the upper imaginary axis, and (c) the first quarter arc at $\zeta\to\infty$. As $\zeta\to\infty$, $K_1(\zeta)\sim e^{-\zeta}$ we see that no zero comes in through wall (c). On the positive imaginary axis, let $\zeta=i\sigma$, we have

$$K_1(i\zeta)=-\tfrac{1}{2}\pi H_1^{(2)}(\sigma)=-\tfrac{1}{2}\pi[J_1(\sigma)-iY_1(\sigma)]\,. \quad \text{(B2)}$$

A number of zeros of $D^{(\pm)}(\zeta,Z)$ enter through wall (b). Their entering positions $\sigma_0^{(\pm)}$ and the corresponding values $Z_0^{(\pm)}$ are the possible solutions of the equations

$$J_1(\sigma)=0 \quad \text{(B3)}$$

and

$$1\mp Z[\sigma^{-2}+\tfrac{1}{2}\pi\sigma^{-1}Y_1(\sigma)]=0\,. \quad \text{(B4)}$$

To check that a zero is entering instead of leaving the first quadrant at σ_0 as Z increases from Z_0, we let

$$\sigma=\sigma_0+\delta\sigma,\quad Z=Z_0+\delta Z\,, \quad \text{(B5)}$$

and examine whether

$$\mathrm{Im}(\delta\sigma/\delta Z)_{(\pm)}^{-1}>0$$

[i.e., $\mathrm{Im}(\delta\sigma/\delta Z)_{(\pm)}<0$ and so $\mathrm{Re}(\delta\zeta/\delta Z)_{(\pm)}>0$]. Substituting (B5) into the equation $D^{(\pm)}(i\sigma,Z)=0$, we get

$$\mathrm{Im}(\delta\sigma/\delta Z)_{(\pm)}^{-1}=\tfrac{1}{2}\pi(Z_0^{(\pm)})^2(\sigma_0^{(\pm)})^{-2}J_1'(\sigma_0^{(\pm)})\,, \quad \text{(B6)}$$

where $J_1'(\sigma)$ is the first derivative of $J_1(\sigma)$. For all sets of values of $[\sigma_0^{(\pm)},Z_0^{(\pm)}]$, (B6) is indeed positive. On the positive real axis, $D^{(+)}(\sigma,Z)$ is nonzero for all Z, while $D^{(-)}(\zeta,Z)$ has a zero near $Z^{1/2}$ for $Z\gg1$. This zero stays on the real axis. No zero comes in through wall (a).

APPENDIX C

From Eq. (4.1), we have, in coordinate space

$$\frac{\partial^2}{\partial(x^2)^2}B^{(\pm)}(x^2)=-\frac{ig^2}{m^2}\frac{\partial^2}{\partial(x^2)^2}\bar{\Delta}_F(x^2)\times\left[1\mp\frac{4ig^2}{m^2}\bar{\Delta}_F(x^2)\right]^{-1}. \quad \text{(C1)}$$

Define a function $B_0(x^2)$ by

$$B_0(x^2)=-\frac{ig^2}{m^2}\bar{\Delta}_F(x^2)\,, \quad \text{(C2)}$$

then the difference

$$\frac{\partial^2}{\partial(x^2)^2}[B^{(\pm)}(x^2)-B_0(x^2)]=\pm\frac{4g^4}{m^4}\bar{\Delta}_F(x^2)\frac{\partial^2}{\partial(x^2)^2}\bar{\Delta}_F(x^2)\left[1\mp\frac{4ig^2}{m^2}\bar{\Delta}_F(x^2)\right]^{-1} \quad \text{(C3)}$$

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[4] Bateman Manuscript Project, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, p. 93.

can be transformed to q space as

$$B^{(\pm)}(x^2)-B_0(x^2)$$
$$=\mp\frac{64g^4}{m^4}\frac{\partial^2}{\partial(q^2)^2}\int d^4x e^{iqx}\bar{\Delta}_F(x^2)$$
$$\times\frac{\partial^2}{\partial(x^2)^2}\bar{\Delta}_F(x^2)\left[1\mp\frac{4ig^2}{m^2}\bar{\Delta}_F(x^2)\right]^{-1}. \quad \text{(C4)}$$

We again use the reduction formula of Feinberg and Pais, and split the integrals (C4) into Bessel integrals and contour integrals. Following the same argument as given in Sec. 4, we can show that the contour integral for $[B^{(+)}(q^2)-B_0(q^2)]$ is zero and that for $[B^{(-)}(q^2)-B_0(q^2)]$ is equal to the residue at the real axis. However,

$$\lim_{q\to 0} q^2[B^{(-)}(q^2)-B_0(q^2)]_C=0.$$

The Bessel integrals are, for $q^2>0$,

$$\mp\frac{16ig^4}{\pi^2m^4}\int_0^\infty \frac{\partial^2}{\partial(q^2)^2}[q^{-1}J_1(q\eta)]\bar{\Delta}_F(\eta)$$
$$\times\frac{\partial^2}{\partial(\eta^2)^2}\bar{\Delta}_F(\eta)\left[1\mp\frac{4ig^2}{m^2}\bar{\Delta}_F(\eta)\right]^{-1}\eta^2 d\eta. \quad \text{(C5)}$$

When $q\to 0$, these integrals diverge logarithmically. Therefore

$$\lim_{q\to 0} q^2[B^{(\pm)}(q^2)-B_0(q^2)]=0 \quad \text{(C6)}$$

for all Λ. And it follows that

$$\lim_{q\to 0} B^{(\pm)}(q^2)q^2=\lim_{q\to 0} B_0(q^2)q^2=-ig^2/m^2. \quad \text{(C7)}$$

APPENDIX D

In this Appendix we study the iterative solution of (5.8). For this purpose, define $T^{(n)}(\eta)$ for $\eta>0$ recurrently as follows:

$$T^{(0)}(\eta)=\tfrac{1}{4} \quad \text{(D1)}$$

and

$$\left(\frac{d^2}{d\eta^2}+\frac{3}{\eta}\frac{d}{d\eta}\right)T^{(n)}(\eta)=-\Lambda\eta^{-1}K_1(\Lambda\eta)T^{(n-1)}(\eta) \quad \text{(D2)}$$

for $n\geq 1$. With (D2), the following boundary conditions are used: near $\eta=0$

$$T^{(n)}(\eta)=o(\eta^{-\alpha}) \quad \text{(D3)}$$

for all $\alpha>0$; and as $\eta\to\infty$

$$T^{(n)}(\eta)\sim a_0^{(n)}\eta^{-2}, \quad \text{(D4)}$$

where $a_0^{(n)}$ is independent of η. Let

$$S_0(z)=\sum_1^\infty a_0^{(n)}z^n \quad \text{(D5)}$$

and

$$\Omega_0(\eta,z)=\sum_1^\infty T^{(n)}(\eta)z^n, \quad \text{(D6)}$$

when the series converge, then

$$T^{(\pm)}=\mp\Omega_0(\eta,\mp Z) \quad \text{(D7)}$$

and

$$a_0^{(\pm)}=\mp S_0(\mp Z). \quad \text{(D8)}$$

We are interested in the radii of convergence of the series (D5–6).

It is convenient to use the variable

$$\tau=\tfrac{1}{2}\Lambda\eta. \quad \text{(D9)}$$

Then $T^{(n)}$ is determined for $n\geq 1$ by the differential equation

$$\left(\frac{d^2}{d\tau^2}+\frac{3}{\tau}\frac{d}{d\tau}\right)T^{(n)}(\tau)=-2\tau^{-1}K_1(2\tau)T^{(n-1)}(\tau), \quad \text{(D10)}$$

together with the boundary conditions

$$T^{(n)}(\tau)=o(\tau^{-\alpha}) \quad \text{(D11)}$$

near $\tau=0$ for all $\alpha>0$, and as $\tau\to\infty$

$$T^{(n)}(\tau)\sim a^{(n)}\tau^{-2}. \quad \text{(D12)}$$

Clearly

$$a^{(n)}=\tfrac{1}{4}\Lambda^2 a_0^{(n)}. \quad \text{(D13)}$$

Let

$$S(z)=\sum_1^\infty a^{(n)}z^n=\tfrac{1}{4}\Lambda^2 S_0(z) \quad \text{(D14)}$$

and

$$\Omega(\tau,z)=\sum_1^\infty T^{(n)}(\tau)z^n=\Omega_0. \quad \text{(D15)}$$

We proceed to study the convergence of $S(z)$ and $\Omega(\tau,z)$.

First, (D10) may be solved to give

$$T^{(n)}(\tau)=\tau^{-2}\int_0^\infty d\tau' K_1(2\tau')T^{(n-1)}(\tau')\min(\tau^2,\tau'^2). \quad \text{(D16)}$$

Thus

$$T^{(n)}(\tau)>0 \quad \text{(D17)}$$

for all τ and all n. Hence, the radii of convergence may be found by estimating $K_1(2\tau)$. We make the following particularly convenient choice. Let

$$F(\tau)=\begin{cases}\tau^{-2}, & \text{for } \tau\leqslant 1,\\ \tau^{-4}, & \text{for } \tau\geqslant 1,\end{cases} \quad \text{(D18)}$$

then

$$2\tau^{-1}K_1(2\tau)<F(\tau). \quad \text{(D19)}$$

Analogous to $T^{(n)}(\tau)$, define $b^{(n)}(\tau)$ recurrently as follows:

$$b^{(0)}(\tau)=\begin{cases}1, & \text{for } \tau<1,\\ 0, & \text{for } \tau\geqslant 1;\end{cases} \tag{D20}$$

and, for $n\geq 1$, $b^{(n)}(\tau)$ is determined by the differential equation

$$\left(\frac{d^2}{d\tau^2}+\frac{3}{\tau}\frac{d}{d\tau}\right)b^{(n)}(\tau)=-F(\tau)b^{(n-1)}(\tau), \tag{D21}$$

together with the boundary conditions that for τ small

$$b^{(n)}(\tau)=o(\tau^{-\alpha}) \tag{D22}$$

for all $\alpha>0$, and that for $\tau\to\infty$

$$b^{(n)}(\tau)\sim\beta^{(n)}\tau^{-2}. \tag{D23}$$

Also let

$$V(z)=\sum_1^\infty \beta^{(n)}z^n \tag{D24}$$

and

$$W(\tau,z)=\sum_1^\infty b^{(n)}(\tau)z^n. \tag{D25}$$

Similar to (D16), (D21) may be solved to give

$$b^{(n)}(\tau)=\tfrac{1}{2}\tau^{-2}\int_0^\infty d\tau'\tau'F(\tau')b^{(n-1)}(\tau')\min(\tau^2,\tau'^2)>0. \tag{D26}$$

Moreover, a somewhat tedious calculation gives

$$b^{(1)}(\tau)>T^{(1)}(\tau). \tag{D27}$$

Hence, by (D19), (D16), and (D26)

$$b^{(n)}(\tau)>T^{(n)}(\tau) \tag{D28}$$

for all $n\geqslant 1$. Therefore, if (D24–25) converge, so do (D14–15).

If (D25) converges, then W satisfies

$$\left(\frac{d^2}{d\tau^2}+\frac{3}{\tau}\frac{d}{d\tau}\right)W(\tau,z)\begin{aligned}&=-z\tau^{-2}[1+W(\tau,z)], &&\text{for } \tau<1,\\ &=-z\tau^{-4}W(\tau,z), &&\text{for } \tau>1.\end{aligned} \tag{D29}$$

Equation (D29) can be solved in terms of Bessel functions. Indeed, this is the motivation for the choice (D18). The result is

$$V(z)=\tfrac{1}{2}D^{-1}[1-(1-z)^{1/2}], \tag{D30}$$

and

$$W(\tau,z)=\begin{cases}-1+\tau^{-1+\sqrt{(1-z)}}D^{-1}[z^{-1/2}J_1(\sqrt{z})+J_1'(\sqrt{z})], & \text{for } \tau\leqslant 1,\\ D^{-1}[1+(1-z)^{1/2}]^{-1}\tau^{-1}z^{1/2}J_1(\tau^{-1}\sqrt{z}), & \text{for } \tau\geqslant 1,\end{cases} \tag{D31}$$

where

$$D=(1-z)^{1/2}z^{-1/2}J_1(\sqrt{z})+J_1'(\sqrt{z}). \tag{D32}$$

To ascertain the analytic behavior of W in the variable z, we need to find the zeros of D. Let

$$f(z,\alpha)=(1-z)^{1/2}z^{-1/2}J_1(\sqrt{z})+\alpha J_1'(\sqrt{z}), \tag{D33}$$

where $\alpha\geqslant 0$. Then, for each α, $f(z,\alpha)$ is analytic in the z plane with a cut along the real axis from 1 to ∞. In particular,

$$D=f(z,1). \tag{D34}$$

When $\alpha=0$, $f(\alpha,0)$ has no zero in the cut plane, but has zeros along the cut, it may be verified that as α increases from zero, these zeros all recede from the cut plane. Moreover, no zero can appear through either this cut or from infinity as α increases. Thus $f(z,\alpha)$, and in particular, D has no zero in the cut plane.

Consequently, for each τ, the right-hand sides of (D30–31) each have a radius of convergence 1 in z when expanded into a power series. Thus, for $|z|<1$, the right-hand sides of (D5–6) converge. It can also be shown that the right-hand sides of (D5–6) diverge when $|z|>1$, but this statement does not seem to be useful.